Ex. 3.7

$f : X \longrightarrow Y = \operatorname{Spec} B$ $\qquad f^{-1}(Y) = \bigcup_{i=1}^{n} V_i \qquad V_i \cong \operatorname{Spec} A_i$

$f_i : V_i \longrightarrow Y \qquad \operatorname{Spec} A_i \longrightarrow \operatorname{Spec} B \Rightarrow \eta \leftrightarrow (0) \subset B$

$K_Y := \mathcal{O}_{Y,\eta} = B_{(0)} = \operatorname{Frac} B.$

$\Downarrow$

$k(\eta) = K_Y$

$f_i^{-1}(\eta) = \operatorname{Spec} A_i \otimes_B K_Y.$

$$\begin{tikzcd}
f^{-1}(\eta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} A_i \arrow[d] \\
\operatorname{Spec} K_Y \arrow[r] & \operatorname{Spec} B
\end{tikzcd}$$

$A_i \otimes_B K_Y = A_i \otimes_B \mathcal{O}_{Y,\eta} = A_i \otimes_B B_{(0)}$

$f_i^{-1}(\eta)$ is finite $\Rightarrow A_i \otimes_B K_Y$ is an Artin ring.
(Atiyah-McDonald)

$\Rightarrow$ finite dim'l vector space $/K_Y$

it is also an integral domain

$\Rightarrow$ field $\Rightarrow = \boxed{\operatorname{Frac} A_i} = A_i \otimes_B K_Y$

$\Rightarrow f_i^{-1}(\eta) = \{\eta_{V_i}\}$

$\operatorname{Frac} A_i = K_X$

$\eta_{V_i} = \eta_{V_j} = \eta_X$

$$K_X = \mathrm{Frac} A_i \quad \text{for all } i$$
$$= A_i \otimes_B K_Y \quad \text{finite}/K_Y$$

$$\operatorname{Spec} A_i \xrightarrow{f_i} \operatorname{Spec} B \qquad \eta_X \longmapsto \eta_Y$$

$$\Updownarrow$$

$$(f_i^\#)^{-1}((0)) = (0)$$

$$\begin{array}{ccc} A_i & \xleftarrow{f_i^\#} & B \\ \downarrow & & \downarrow \\ \mathrm{Frac} A_i = K_X & \underset{\text{finite}}{\longleftarrow} & K_Y = \mathrm{Frac} B \end{array} \Rightarrow A_i \text{ is finite}/B$$

$$f_1 : U_1 \longrightarrow Y \qquad X \setminus U_1 \not\ni \eta_X \qquad X \setminus U_1 = \bigcup_{i=2}^{n} ((X \setminus U_1) \cap U_i)$$

$$f(X \setminus U_1) \underset{?}{\subset} \text{closed in } Y$$

$(X \setminus U_1) \cap U_i$ closed in $U_i = \operatorname{Spec} A_i \longrightarrow \operatorname{Spec} B$. finite $\Rightarrow$ closed

$$\Rightarrow f(X \setminus U_1) = \bigcup_{i=2}^{n} f((X \setminus U_1) \cap U_i) \text{ is closed}$$

(going-up theorem and Lemma 4.5)

Choose $\operatorname{Spec} B[g^{-1}] \subset Y \setminus f(X \setminus U_1)$, $f^{-1}(\operatorname{Spec} B[g^{-1}])$
$= \operatorname{Spec} A_1[g^{-1}]$

Ex. 3.10 (a) $f : X \to Y \ni y$ $\quad f^{-1}(y)$

$\qquad\qquad\qquad\quad\ \cup$

$f : f^{-1}(\operatorname{Spec} B) \longrightarrow \operatorname{Spec} B \ni y$

replace Y with $\operatorname{Spec} B$, X with $f^{-1}(\operatorname{Spec} B)$

$X = \bigcup_{i \in I} U_i \qquad U_i = \operatorname{Spec} A_i$

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ \operatorname{Spec} k(y) \arrow[r] & Y = \operatorname{Spec} B. \end{tikzcd}$$

$X_y \cap U_i = \operatorname{Spec} A_i \otimes_B k(y)$

$k(y) = B_{\mathfrak{q}} / \mathfrak{q} B_{\mathfrak{q}}.$

$y \hookrightarrow \mathfrak{q} \subset B \qquad \mathcal{O}_{Y,y} = B_{\mathfrak{q}}.$

$X_y \cap U_i = \operatorname{Spec} A_i \otimes_{B.} B_{\mathfrak{q}} / \mathfrak{q} B_{\mathfrak{q}}$ $\qquad A_i \longleftarrow B \supset \mathfrak{q}$

$f^{-1}(y) \cap U_i = \{$prime ideals of A_i which pull back to $\mathfrak{q}$ in $B\}$.

$\{$prime ideals of $A_i \otimes_B B_{\mathfrak{q}}\} = \{$prime ideals of A_i whose pull-back to B is contained in $\mathfrak{q}\}$

$$\left\{ \text{prime ideals of } \frac{A_i \otimes B_{\mathfrak{q}}}{A_i \otimes \mathfrak{q} B_{\mathfrak{q}}} \right\} = \left\{ \begin{array}{l} \text{prime ideals of } A_i \text{ whose} \\ \text{pull-back to } B \text{ is contained} \\ \text{in } \mathfrak{q}\text{, and contains } \mathfrak{q} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \text{prime ideals of } A_i \text{ whose} \\ \text{pull-back to } B \text{ is } \mathfrak{q} \end{array} \right\}$$

The topology is the same because on both sides, the closed sets are the sets of prime ideals containing a fixed ideal.